# SEGMENTS IN BALL PACKINGS

Dedicated to Professor Rogers on the occasion of his 80th birthday

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*Abstract.* Denote by $B^n$ the $n$-dimensional unit ball centred at $\mathbf{o}$. It is known that in every lattice packing of $B^n$ there is a cylindrical hole of infinite length whenever $n \geq 3$. As a counterpart, this note mainly proves the following result: *For any fixed $\epsilon$, $\epsilon > 0$, there exist a periodic point set $P(n, \epsilon)$ and a constant $c(n, \epsilon)$ such that $B^n + P(n, \epsilon)$ is a packing in $R^n$, and the length of the longest segment contained in $R^n \setminus \{\mathrm{int}(\epsilon B^n) + P(n, \epsilon)\}$ is bounded by $c(n, \epsilon)$ from above.* Generalizations and applications are presented.

§*1. Introduction.* Let $B^n$ be the $n$-dimensional unit ball centred at the origin $\mathbf{o}$ of $R^n$, let $P$ be a set of discrete points such that $B^n + P$ is a packing, let $s(B^n, P)$ be the length of the longest segment contained in $R^n \setminus \{\mathrm{int}(B^n) + P\}$, and define

$$s(B^n) = \inf_P \{s(B^n, P)\},$$

where the infimum is taken over all the sets $P$ such that $B^n + P$ is a packing.

Let $s^*(B^n)$ be the corresponding number when $P$ is a *packing lattice.* It follows from the results of Heppes [6], Hortobágyi [8], Horváth [9], and Horváth and Ryškov [10] that, when $n \geq 3$,

$$s^*(B^n) = \infty. \tag{1}$$

In fact, they proved that in every lattice ball packing $B^n + \Lambda$ there is a *cylindrical hole* of infinite length whenever $n \geq 3$.

In $R^2$, by elementary arguments and routine computations it can be deduced that

$$s(B^2) = s^*(B^2) = 2(\sqrt{3} - 1),$$

where the equality can be realized by the *hexagonal disk packing.*

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For convenience, we call a straight line with one end extending to infinity a *light ray*. Let $l(B^n)$ be the smallest number of non-overlapping unit balls outside of $B^n$ such that every light ray starting from $B^n$ can be blocked by them. This number has been studied by Bárány, Böröczky, Henk, Leader, Soltan, Talata, Zong, and others (see Martini and Soltan [11] and Zong [16]). It is proved that

$$2^{cn^2(1+o(1))} \le l(B^n) \le 2^{c'n^2(1+o(1))},$$

where $c$ and $c'$ are some positive constants. Clearly, this result implies

$$s(B^n) \ge 2^{cn(1+o(1))}.$$

Based on this observation, Zong [16] proposed the following problem: *Does there exist a constant c such that*

$$s(B^n) = 2^{cn(1+o(1))}? \tag{2}$$

Such a result will imply all the known results about blocking light rays.

We call a point set $P$ *periodic* if it can be expressed as a sum of a finite point set and a lattice. This note mainly proves the following result:

Theorem 1. *For any fixed $\epsilon$, $\epsilon > 0$, there exist a periodic point set $P(n, \epsilon)$ and a constant $c(n, \epsilon)$ such that $B^n + P(n, \epsilon)$ is a packing in $R^n$ and*

$$s(\epsilon B^n, P(n, \epsilon)) \le c(n, \epsilon). \tag{3}$$

Although this result is far from proving (2), it shows the fundamental difference between $s(B^n)$ and $s^*(B^n)$ by comparing (3) with (1). Similar to $s(B^n)$, one can define $s(K)$ for any $n$-dimensional *convex body* $K$. Then we have the following general result:

Theorem 2. *For any fixed $n$-dimensional convex body $K$ and any fixed $\epsilon$, $\epsilon > 0$, there exist a periodic point set $P(K, \epsilon)$ and a constant $c(K, \epsilon)$ such that $K + P(K, \epsilon)$ is a packing and*

$$s(\epsilon K, P(K, \epsilon)) \le c(K, \epsilon).$$

For more information about ball packings, we refer to Rogers [12] and Zong [16].

§*2. Proofs of the theorems.* As usual, we denote by $\|X, Y\|$ the Euclidean distance between two sets $X$ and $Y$, by $\lambda(X)$ the minimal distance between distinct points of $X$, by $S(\mathbf{x}, t\mathbf{v})$ the segment $\{\mathbf{x}+\theta\mathbf{v} :\ 0 \le \theta \le t\}$, where $\mathbf{v}$ is a unit vector, and by $L(\mathbf{x}, \mathbf{v})$ the half line $\{\mathbf{x} + \theta\mathbf{v} :\ \theta \ge 0\}$. In addition, we write $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \dots, x_{i,n})$, $\mathbf{e}_n = (0, 0, \dots, 0, 1)$, and denote by $Z$ the set of all integers.

To prove theorem 1, by applying induction, we construct two periodic packings of $B^n$ based on the same $(n-1)$-dimensional lattice, with the following properties: The first one, mainly constructed in lemma 1, intersects every long segment in the unit directions $\mathbf{v}$ if the last coordinate $|v_n|$ is not too small. The second packing set, given by induction from theorem 1, intersects every long segment in the remaining unit directions $\mathbf{v}$, i.e., if $|v_n|$ is small. Finally, we have to join these two packing sets in a proper way to produce the packing set with the desired properties.

Lemma 1. *Let $\Lambda$ be a packing lattice of $B^{n-1}$, and let $\alpha$ and $\beta$ be fixed numbers between* 0 *and* 1. *There exist a finite point set $X = \{\mathbf{x}_1 = \mathbf{o}, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ in $R^n = R^{n-1} \oplus R^1$ and a constant $c(\Lambda, n, \alpha, \beta)$ such that*

**1.** $|x_{i+1,n} - x_{i,n}| = 2$ *for* $i = 1, 2, \dots, m-1$.

**2.** $S(\mathbf{x}, c(\Lambda, n, \alpha, \beta)\mathbf{v}) \cap (\mathrm{int}(\alpha B^n) + \Lambda + X) \neq \emptyset$ *whenever* $x_n = 0$ *and* $v_n \ge \beta$.

*Proof.* Since $\alpha B^{n-1} + \Lambda$ is a lattice packing in $R^{n-1}$, there exists a finite point set $X^* \subset R^{n-1}$ of minimal *cardinality* such that

$$R^{n-1} \subseteq \tfrac{1}{2}\mathrm{int}(\alpha B^{n-1}) + \Lambda + X^*. \tag{4}$$

In other words, the system on the right-hand side of (4) is a covering of $R^{n-1}$. For convenience, we write

$$k = \mathrm{card}\{X^*\}.$$

This number depends only on $\Lambda$ and $\alpha$. Then we take

$$\gamma = \alpha\beta/4k \tag{5}$$

and choose a finite point set $V = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_l\}$ of minimal cardinality from

$$\Omega = \{\mathbf{v} \in \partial(B^n) :\ v_n \ge \beta\}$$

such that the sets

$$\Omega_i = \Omega \cap (\mathrm{int}(\gamma B^n) + \mathbf{v}_i)$$

form a *covering* in $\Omega$. Here $l$ depends only on $\Lambda$, $n$, $\alpha$ and $\beta$.

$\alpha B^n + \Lambda + p(\mathbf{y}, i, j)$

$p(\mathbf{y}, i, j)$

$L(\mathbf{x}, \mathbf{v})$

$\frac{1}{2}\alpha B^{n-1} + \Lambda + \mathbf{y}$

$\mathbf{v}$

$\mathbf{v}_i$

$\mathbf{y}$

Figure 1

Now we consider the segments in directions $\mathbf{v} \in \Omega_i$. For any point $\mathbf{y}$ we denote by $p(\mathbf{y}, i, j)$ the intersection of $L(\mathbf{y}, \mathbf{v}_i)$ and the hyperplane $R^{n-1} + \mathbf{y} + 2(j-1)\mathbf{e}_n$. By (5) and routine computations based on elementary geometry it follows that

$$L(\mathbf{y}, \mathbf{v}) \cap (\tfrac{1}{2}\mathrm{int}(\alpha B^n) + p(\mathbf{y}, i, j)) \neq \emptyset$$

whenever $1 \leq j \leq k$ and $\mathbf{v} \in \Omega_i$. Therefore (see Figure 1) we have

$$L(\mathbf{x}, \mathbf{v}) \cap (\mathrm{int}(\alpha B^n) + p(\mathbf{y}, i, j)) \neq \emptyset \tag{6}$$

whenever $1 \leq j \leq k$, $\mathbf{v} \in \Omega_i$ and

$$\mathbf{x} \in \tfrac{1}{2}\mathrm{int}(\alpha B^{n-1}) + \mathbf{y}.$$

Taking

$$X_i = \{p(\mathbf{y}_j + 2(i-1)k\mathbf{e}_n, i, j) : \ \mathbf{y}_j \in X^*\},$$

the system $B^n + \Lambda + X_i$ is a packing, and $X_i$ (as a part) satisfies the first assertion of our lemma. In addition, by (4) it follows that $\frac{1}{2}\mathrm{int}(\alpha B^{n-1}) + \Lambda + X^* + 2(i-1)k\mathbf{e}_n$ is a covering in $R^{n-1} + 2(i-1)k\mathbf{e}_n$. Thus, by (6) and simple arguments we have

$$S(\mathbf{x}, (2kl/\beta)\mathbf{v}) \cap (\mathrm{int}(\alpha B^n) + \Lambda + X_i) \neq \emptyset$$

whenever $x_n = 0$ and $\mathbf{v} \in \Omega_i$. Clearly $2kl/\beta$ depends only on $\Lambda$, $n$, $\alpha$ and $\beta$.

To deal with all the segments in directions $\mathbf{v} \in \Omega$ we take

$$X = \bigcup_{i=1}^{l} X_i$$

and

$$c(\Lambda, n, \alpha, \beta) = 2kl/\beta.$$

Since the cap system $\Omega_i$ covers $\Omega$, the two assertions of lemma 1 are satisfied. The proof is done. $\square$

*Proof of theorem 1.* Clearly theorem 1 is true when $n = 1$. Assume the theorem is true in $R^{n-1}$. Without loss of generality, we make the following assumption:

Assumption. *For any fixed $\alpha$, $\alpha > 0$, there exist a lattice $\Lambda$, a finite point set $Y$, and a constant $c'(n-1, \alpha)$ such that $3B^{n-1} + \Lambda + Y$ is a packing in $R^{n-1}$ and*

$$s(\tfrac{1}{2}\alpha B^{n-1}, \Lambda + Y) \leq c'(n-1, \alpha). \tag{7}$$

We proceed to prove the $n$-dimensional case by induction.

If theorem 1 is true for $\epsilon$, $\epsilon > 0$, then it is also true for any number larger than $\epsilon$. Thus, without loss of generality, let $\alpha = \epsilon^2 = 1/M^2$ for some large integer $M$. Considering the corresponding packing $\alpha B^n + \Lambda + Y$ in $R^n$ and its intersections with hyperplanes $R^{n-1} + \theta \mathbf{e}_n$, $0 \leq \theta \leq \alpha/2$. By (7) and routine computations based on elementary geometry it follows that

$$S(\mathbf{x}, 2c'(n-1, \alpha)\mathbf{v}) \cap (\mathrm{int}(\alpha B^n) + \Lambda + Y) \neq \emptyset \tag{8}$$

whenever $x_n = 0$ and $|v_n| \leq \alpha/4c'(n-1, \alpha)$. Since $3B^{n-1} + \Lambda + Y$ is a packing in $R^{n-1}$ we have

$$\lambda(\Lambda + Y) \geq 6. \tag{9}$$

Without loss of generality, we assume that $\mathbf{o}$ and $\mathbf{u}$ are two points of $\Lambda + Y$ such that

$$\|\mathbf{o}, \mathbf{u}\| = \lambda(\Lambda + Y).$$

Then we define

$$\mathbf{p}(j, z) = \tfrac{(j-1)}{3}\epsilon \mathbf{u} + ((j-1)\epsilon^2 + 2(z-1)\epsilon)\mathbf{e}_n$$

and

$$P = \{\Lambda + Y + \mathbf{p}(j, z) : \ 1 \leq j \leq 2M; \ z \in Z\}.$$

Observing the projection of $P$ onto $R^{n-1}$ it follows that

$$\lambda(P) \geq 2\epsilon. \tag{10}$$

In addition, by considering the intersections

$$(\alpha B^n + P) \cap (R^{n-1} + \theta \mathbf{e}_n)$$

and applying (8) we have

$$S(\mathbf{x}, 4c'(n-1, \alpha)\mathbf{v}) \cap (\mathrm{int}(\alpha B^n) + P) \neq \emptyset \tag{11}$$

whenever $|v_n| \leq \alpha/4c'(n-1, \alpha)$.

Let $X$ be a suitable set of $m$ points and $c(\Lambda, n, \alpha, \beta)$ be a suitable number of lemma 1 corresponding to the lattice $\Lambda$ of the assumption, $\alpha = \epsilon^2$, and $\beta = \alpha/4c'(n-1, \alpha)$, and write

$$\Lambda_n = \{\Lambda + 2zm\mathbf{e}_n : \ z \in Z\}.$$

Clearly, $c(\Lambda, n, \alpha, \beta)$ can be determined by $n$ and $\alpha$. So we write

$$c(\Lambda, n, \alpha, \beta) = c_1(n, \alpha).$$

Then $B^n + \Lambda_n + X$ is a packing in $R^n$, and

$$S(\mathbf{x}, 2c_1(n, \alpha)\mathbf{v}) \cap (\mathrm{int}(\alpha B^n) + \Lambda_n + X) \neq \emptyset \tag{12}$$

whenever $|v_n| \geq \beta$. By the definitions of $P$ and $\Lambda_n$, we have

$$P = \Lambda_n + Y + P_1,$$

where

$$P_1 = \{\mathbf{p}(j, z) : \ 1 \leq j \leq 2M; \ 1 \leq z \leq m/\epsilon = mM\}.$$

For any $\mathbf{x}_i \in X$, by the definition of $P$ there are at most four of its points, say $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{p}_4$, that

$$\|\Lambda + \mathbf{x}_i, \Lambda + Y + \mathbf{p}_j\| < 2\epsilon.$$

By (9) and routine arguments, there is a point $\mathbf{y} \in Y$ such that

$$\|\Lambda + \mathbf{x}_i, \Lambda + Y \setminus \{\mathbf{y}\} + \mathbf{p}_j\| \geq \|\mathbf{u}\| - 4\epsilon > 5. \tag{13}$$

In this case, if

$$\mathbf{p}_j = a_j\mathbf{u} + b_j\mathbf{e}_n,$$

we replace $\mathbf{p}_j$ by

$$\mathbf{p}'_j = \tfrac{2M+j+2}{3}\epsilon\mathbf{u} + b_j\mathbf{e}_n. \tag{14}$$

Then it follows from (13), (14) and routine computations that

$$\begin{aligned}\|\Lambda + \mathbf{x}_i, \Lambda + Y + \mathbf{p}'_j\| \ &\geq \ \min\{\|\Lambda + \mathbf{x}_i, \Lambda + \mathbf{y} + \mathbf{p}'_j\|, \\ &\qquad\qquad \|\Lambda + \mathbf{x}_i, \Lambda + Y \setminus \{\mathbf{y}\} + \mathbf{p}'_j\|\} \\ &\geq \ 2\epsilon.\end{aligned}$$

Considering the points of $X$ one by one, we obtain a new point set

$$P' = \Lambda_n + Y + P'_1.$$

Observing the projections of $P$ and $P'$ to $R^{n-1}$, it follows that the set $P'$ satisfies (10). In fact, the modification process is just moving the corresponding layers of small balls in the direction of $\mathbf{u}$. Thus the set $P'$ satisfies also (11). Consequently, defining

$$P(\alpha) = \Lambda_n + X \cup \{Y + P'_1\},$$

we have

$$\lambda(P(\alpha)) \geq 2\epsilon.$$

On the other hand, taking

$$c_2(n, \alpha) = \max\{8c'(n-1, \alpha), 4c_1(n, \alpha)\},$$

by (11) and (12) we have

$$s(\alpha B^n, P(\alpha)) \leq c_2(n, \alpha).$$

Then, by taking

$$P(n, \epsilon) = \tfrac{1}{\epsilon}P(\epsilon^2)$$

and

$$c(n, \epsilon) = \tfrac{1}{\epsilon}c_2(n, \epsilon^2),$$

theorem 1 is proved. $\square$

Remark 1. *In* 1967*, Böröczky* [1] *proved in* $R^3$ *that there is a ball packing* $B^3 + P$ *between two hyperplanes* $H_1$ *and* $H_2$ *such that any segment* $[\mathbf{x}, \mathbf{y}]$*,*

$\mathbf{x} \in H_1$ *and* $\mathbf{y} \in H_2$*, intersects* $\text{int}(B^3)+P$*. Clearly, as a corollary of theorem 1, this assertion is true in every dimension.*

For any fixed convex body $K$ there are two positive numbers $r_1$ and $r_2$ such that
$$r_1 B^n \subseteq K \subseteq r_2 B^n.$$
Thus, theorem 2 follows from theorem 1 as a corollary.

§*3. Applications and related problems.* Suppose $B^n + P$ is a packing. Let $w_i(B^n, P)$ be the maximal $i$-th *quermassintegral* of a convex set contained in
$$R^n \setminus \{\text{int}(B^n) + P\},$$
and define
$$w_i(B^n) = \inf_P \{w_i(B^n, P)\}.$$
Similarly, denote by $w_i^*(B^n)$ the corresponding numbers when $P$ is a packing lattice. As a consequence of theorem 1 and the result just below (1) we have the following result about $w_i(B^n)$ and $w_i^*(B^n)$:

Theorem 3. *For each* $i$ *with* $0 \leq i \leq n-1$ *we have*
$$w_i^*(B^n) = \infty$$
*and*
$$w_i(B^n) \leq c_{n,i}, \tag{15}$$
*where* $c_{n,i}$ *is a constant depends on* $n$ *and* $i$*.*

Let $K_{n,i}$ be a convex set such that there is a packing $B^n + P$,
$$K_{n,i} \subseteq R^n \setminus \{\text{int}(B^n) + P\},$$
and
$$W_i(K_{n,i}) = w_i(B^n) = w_i(B^n, P),$$
where $W_i(\cdot)$ indicates the $i$-th quermassintegral. It is clear that $K_{n,i}$ is a polytope.

Problem 1. *Decide the shapes of these extreme polytopes.*

For arbitrary convex body $K$, applying theorem 2, similar inequality as (15) can be proved. In addition, one can ask question similar to problem 1.

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